



Multi-term fractional oscillation integro-differential equations

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Abstract

In this paper we study solvability of multi-term Caputo and Riemann-Liouville fractional oscillation integro-differential equations. We show that these equations have unique solutions in the space of functions with square average power growth and derive the solutions in explicit forms.

Keywords Functions with square average power growth (primary) · Laplace transform · Caputo fractional derivative · Riemann-Liouville fractional derivative · Fractional oscillation integro-differential equation

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1 Introduction

Recently fractional integro-differential equations have arisen in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electro-dynamics of complex medium, polymer rheology. The fractional integro-differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes.

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Examples include viscoelastic behavior [13], chaotic neuron model [22], long-time memory in financial time series via fractional Langevin equations [24], transport of passive tracers carried by fluid flow in a porous medium in groundwater hydrology [27], etc. Further examples can be found, for example, in [12, 19].

Consider now the following multi-term Caputo and Riemann-Liouville fractional integro-differential initial value problems

$$\begin{aligned} {}^C\partial_t^{\alpha_0} f(t) + \sum_{j=1}^n a_j {}^C\partial_t^{\alpha_j} f(t) + kf(t) + \int_0^t g(t-\tau)f(\tau)d\tau &= h(t), \\ f^{(i)}(0+) &= f_i, \quad i = 0, 1, \dots, m-1, \end{aligned} \quad (1.1)$$

$$\begin{aligned} D_{0+}^{\alpha_0} f(t) + \sum_{j=1}^n a_j D_{0+}^{\alpha_j} f(t) + kf(t) + \int_0^t g(t-\tau)f(\tau)d\tau &= h(t), \\ I_{0+}^{m-i-\alpha_0} f(0+) &= f_i, \quad i = 0, 1, \dots, m-1, \end{aligned} \quad (1.2)$$

$$0 < \alpha_n < \dots < \alpha_1 < \alpha_0, \quad \alpha_0 \in (m-1, m], m \in \mathbb{N}, m \geq 1,$$

where $k, a_1, a_2, \dots, a_n \in \mathbb{R}_+$, $g, h \in L^1(\mathbb{R}_+)$ are given, and f is the unknown. Here D_{0+}^{α} is the Riemann-Liouville fractional derivative [10]

$$\begin{aligned} D_{0+}^{\alpha} f(t) &= \frac{d^m}{dt^m} I_{0+}^{m-\alpha} f(t), \quad 0 \leq \alpha < m \in \mathbb{N} \\ I_{0+}^{m-\alpha} f(t) &= \int_0^t \frac{(t-\tau)^{m-\alpha-1}}{\Gamma(m-\alpha)} f(\tau) d\tau, \end{aligned} \quad (1.3)$$

and ${}^C\partial_t^{\alpha}$ is the Caputo fractional derivative [10]

$$\begin{aligned} {}^C\partial_t^{\alpha} f(t) &= \int_0^t \frac{(t-\tau)^{m-\alpha-1}}{\Gamma(m-\alpha)} f^{(m)}(\tau) d\tau, \quad m-1 < \alpha < m \in \mathbb{N}, \\ {}^C\partial_t^m f(t) &= f^{(m)}(t), \end{aligned} \quad (1.4)$$

and $f(0+) = \lim_{0 < t \rightarrow 0} f(t)$.

Multi-term fractional integro-differential equations (1.1) and (1.2) play an important role in modeling of motion of a rigid plate immersed in a Newtonian fluid. One of the important special cases of (1.1), Bagley-Torvik equation with fractional-order derivative illustrates the motion of real physical systems in a Newtonian fluid [28]. A special case of (1.2), Kelvin-Voigt model is of the most frequent use in structural dynamics [26]. Some special cases of equations (1.1) and (1.2) and their numerical solutions on a finite interval $(0, T)$ have been considered in [4–6]. One can find some recent works on multi-term fractional integro-differential equations in [2, 3, 7, 11, 16, 17].

In [31], Eqs. (1.1) and (1.2) have been studied for the first time on the whole infinite interval $(0, \infty)$. Namely, following [31], by $BSA_p(\mathbb{R}_+)$, $p \geq 0$, we denote the space of Lebesgue measurable functions f on $\mathbb{R}_+$ with square average power growth

$$f \in BSA_p(\mathbb{R}_+) \iff \sup_{T>0} \frac{1}{(T+1)^p} \int_0^T |f(t)|^2 dt < \infty. \quad (1.5)$$

It was shown in [31] that if $\frac{1}{2} < \alpha_0 \leq 1$, and $\|g\|_1 < k$, then the multi-term Caputo fractional integro-differential equation (1.1) has a unique solution f from $BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$, and the multi-term Riemann-Liouville fractional integro-differential equation (1.2) has a unique solution f from $BSA_{2\alpha_0-1}(\mathbb{R}_+)$.

In [30, 31] we have considered only fractional equations (1.1) and (1.2) of order $\alpha_0 \leq 1$, i.e., fractional relaxation processes [21]. In this paper we will study fractional equations (1.1) and (1.2) of order $1 < \alpha_0 \leq 2$, i.e., fractional oscillation processes [8, 21] in $BSA_p(\mathbb{R}_+)$.

2 Fractional Oscillation Processes

It was shown [31] that $f \in BSA_p(\mathbb{R}_+)$ if, and only if, $F(s)$, its Laplace transform, defined by

$$F(s) = (\mathcal{L}f)(s) := \int_0^\infty e^{-st} f(t) dt, \quad (2.1)$$

is holomorphic in the right-half plane $\operatorname{Re} s > 0$, and

$$\sup_{x>0} \left(\frac{x}{x+1} \right)^p \int_{-\infty}^\infty |F(x+iy)|^2 dy < \infty. \quad (2.2)$$

Denote by $\mathcal{L}^{-1}$ the inverse Laplace transform [32]

$$f(t) = (\mathcal{L}^{-1}F)(t) := \frac{1}{2\pi i} \int_{\operatorname{Re} s=d} F(s) e^{st} ds. \quad (2.3)$$

It was proved in [31] that

Corollary 1 *Let $F(s)$ be holomorphic in the right half plane $\operatorname{Re} s > 0$ and $|F(s)| \leq C|s|^{-\alpha}$, $\alpha > \frac{1}{2}$. Then F is the Laplace transform of a function $f \in BSA_{2\alpha-1}(\mathbb{R}_+)$,*

and that

Lemma 1 *Let $f \in BSA_p(\mathbb{R}_+)$ and $g \in L^1(\mathbb{R}_+)$. Then their Laplace convolution*

$$h(t) = (f * g)(t) := \int_0^t f(t-\tau) g(\tau) d\tau \quad (2.4)$$

belongs to $BSA_p(\mathbb{R}_+)$.

We start by showing that

Lemma 2 *Let $1 < \alpha < 2$ and $k > 0$. Then*

$$|s^\alpha + k| > \frac{\sin \frac{\alpha\pi}{2}}{1 + \sin \frac{\alpha\pi}{2}} |s|^\alpha, \quad (2.5)$$

when $\operatorname{Re} s > 0$.

Proof If $\operatorname{Re} s > 0$, then $s^\alpha + k$ is inside the angle $-\frac{\alpha\pi}{2} < \arg(z) < \frac{\alpha\pi}{2}$, shifted to the right by k units. The distance from the origin to the shifted angle is $k \sin \frac{\alpha\pi}{2}$. Then

$$|s^\alpha + k| > k \sin \frac{\alpha\pi}{2} \geq \frac{\sin \frac{\alpha\pi}{2}}{1 + \sin \frac{\alpha\pi}{2}} |s|^\alpha, \quad \operatorname{Re} s > 0,$$

when $|s|^\alpha \leq k(1 + \sin \frac{\alpha\pi}{2})$. If $|s|^\alpha > k(1 + \sin \frac{\alpha\pi}{2})$, then

$$|s^\alpha + k| \geq |s|^\alpha - k > |s|^\alpha - \frac{|s|^\alpha}{1 + \sin \frac{\alpha\pi}{2}} = \frac{\sin \frac{\alpha\pi}{2}}{1 + \sin \frac{\alpha\pi}{2}} |s|^\alpha,$$

that is again (2.5). The lemma is proved. $\square$

Combining [31, Corollary 3.2], Corollary 1, and Lemma 2, we arrive at

Corollary 2 *Let $k > 0$, $0 < \alpha < 2$, $\beta < \alpha - \frac{1}{2}$, then*

$$\mathcal{L}^{-1} \left(\frac{s^\beta}{s^\alpha + k} \right) \in BSA_{2(\alpha-\beta)-1}(\mathbb{R}_+). \quad (2.6)$$

Theorem 1 *Let $1 < \alpha < 2$, $k > 0$, and $h, h' \in L^1(\mathbb{R}_+)$ with $h(0+) = 0$. Then the following Caputo fractional oscillation process*

$${}^C \partial_t^\alpha f(t) + kf(t) = h(t), \quad f(0+) = f_0, \quad f'(0+) = f_1, \quad (2.7)$$

has a unique solution f from $BSA_{2\alpha+1}(\mathbb{R}_+)$.

Proof The solution to the initial value problem (2.7) is well-known (see [18]). Applying the Laplace transform to (2.7) we get

$$F(s) = \frac{s^{\alpha-1} f_0 + s^{\alpha-2} f_1 + H(s)}{s^\alpha + k}. \quad (2.8)$$

Denote

$$U_0(s) = \frac{s^{\alpha-1}}{s^\alpha + k}, \quad U_1(s) = \frac{s^{\alpha-2}}{s^\alpha + k}, \quad U_2(s) = \frac{1}{s^\alpha + k}, \quad (2.9)$$

and $u_0(t)$, $u_1(t)$, $u_2(t)$ - their inverse Laplace transforms. The solution of the initial value problem (2.7) can be written in the form

$$f(t) = f_0 u_0(t) + f_1 u_1(t) + (u_2 * h)(t). \quad (2.10)$$

By Corollary 2 we get

$$u_0(t) \in BSA_1(\mathbb{R}_+), \quad u_1(t) \in BSA_3(\mathbb{R}_+), \quad u_2(t) \in BSA_{2\alpha-1}(\mathbb{R}_+).$$

Since $h, h' \in L^1(\mathbb{R}_+)$ and $h(0+) = 0$, then

$$\begin{aligned} |sH(s)| &= |sH(s) - h(0+)| = |(\mathcal{L}h')(s)| \\ &\leq \int_0^\infty e^{-(\operatorname{Re} s)t} |h'(t)| dt \leq \|h'\|_1, \quad \operatorname{Re} s > 0. \end{aligned}$$

Consequently,

$$\left| \frac{H(s)}{s^\alpha + k} \right| < \frac{(1 + \sin \frac{\alpha\pi}{2}) \|h'\|_1}{\sin \frac{\alpha\pi}{2}} |s|^{-\alpha-1}.$$

By Corollary 2 we get $(u_2 * h)(t) \in BSA_{2\alpha+1}(\mathbb{R}_+)$. Since $2\alpha + 1 > 3 > 2\alpha - 1 > 1$, from (2.10) and [31]

$$BSA_p(\mathbb{R}_+) \subset BSA_q(\mathbb{R}_+), \quad 0 \leq p < q, \quad (2.11)$$

we obtain $f(t) \in BSA_{2\alpha+1}(\mathbb{R}_+)$.

We verify now that $f(t)$, defined by (2.10), satisfies the initial conditions in (2.7). From (2.8) we have $F(s) \sim \frac{f_0}{s}$ as $\operatorname{Re} s \rightarrow \infty$. Therefore, by the Tauberian theorem for the Laplace transform [32]

$$f(t) \sim \frac{At^{p-1}}{\Gamma(p)}, \quad t \rightarrow 0+ \quad \Leftrightarrow \quad F(s) \sim \frac{A}{s^p}, \quad \operatorname{Re} s \rightarrow \infty, \quad p > 0, \quad (2.12)$$

we get $f(t) \rightarrow f_0$ as $t \rightarrow 0+$. Next, as $\operatorname{Re} s \rightarrow \infty$

$$\begin{aligned} (\mathcal{L}f')(s) &= sF(s) - f(0) = \frac{s^\alpha f_0 + s^{\alpha-1} f_1 + sH(s)}{s^\alpha + k} - f_0 \\ &= \frac{s^{\alpha-1} f_1 + sH(s) - kf_0}{s^\alpha + k} \sim \frac{f_1}{s}, \end{aligned}$$

again by the Tauberian theorem (2.12) we obtain $f'(t) \rightarrow f_1$ as $t \rightarrow 0+$. $\square$

Theorem 2 Let $\frac{3}{2} < \alpha < 2$, $k > 0$, and $h \in L^1(\mathbb{R}_+)$. Then the following Riemann-Liouville fractional oscillation process

$$D_{0+}^\alpha f(t) + kf(t) = h(t), \quad I^{2-\alpha} f(0+) = f_0, \quad D^{\alpha-1} f(0+) = f_1, \quad (2.13)$$

has a unique solution f from $BSA_{2\alpha-1}(\mathbb{R}_+)$.

Proof The solution to the initial value problem (2.13) is well-known (see [18]). Applying the Laplace transform to (2.13) we get

$$F(s) = \frac{sf_0 + f_1 + H(s)}{s^\alpha + k}. \quad (2.14)$$

Denote

$$U_3(s) = \frac{s}{s^\alpha + k}, \quad (2.15)$$

and $u_3(t)$ - its inverse Laplace transform. We can write the solution of the initial value problem (2.7) in the form

$$f(t) = f_0 u_3(t) + f_1 u_2(t) + (u_2 * h)(t), \quad (2.16)$$

where $u_2(t)$ is defined in (2.9).

By Lemma 1, and Corollary 2 we get

$$u_2(t) \in BSA_{2\alpha-1}(\mathbb{R}_+), \quad u_3(t) \in BSA_{2\alpha-3}(\mathbb{R}_+), \quad (u_2 * h)(t) \in BSA_{2\alpha-1}(\mathbb{R}_+).$$

Thus, from (2.11) we get $f(t) \in BSA_{2\alpha-1}(\mathbb{R}_+)$.

We verify now that $f(t)$, defined by (2.16), satisfies the initial conditions in (2.13). As $h \in L^1(\mathbb{R}_+)$, then $H(s) \rightarrow 0$ as $\operatorname{Re} s \rightarrow \infty$. Consequently, formula (2.14) yields

$$F(s) \sim \frac{f_0}{s^{\alpha-1}}, \quad \operatorname{Re} s \rightarrow \infty,$$

and by the Tauberian theorem (2.12)

$$f(t) \sim \frac{f_0 t^{\alpha-2}}{\Gamma(\alpha-1)}, \quad t \rightarrow 0+.$$

We have

$$I_{0+}^{2-\alpha} f(t) \sim \frac{f_0 \left(I_{0+}^{2-\alpha} t^{\alpha-2} \right)}{\Gamma(\alpha-1)} = f_0, \quad t \rightarrow 0+.$$

Next,

$$\begin{aligned} (\mathcal{L} D_{0+}^{\alpha-1} f)(s) &= s^{\alpha-1} F(s) - I_{0+}^{2-\alpha} f(0+) = \frac{s^\alpha f_0 + s^{\alpha-1} f_1 + s^{\alpha-1} H(s)}{s^\alpha + k} - f_0 \\ &= \frac{s^{\alpha-1} f_1 + s^{\alpha-1} H(s) - k f_0}{s^\alpha + k} \sim \frac{f_1}{s}, \quad \operatorname{Re} s \rightarrow \infty. \end{aligned}$$

Consequently, again by the Tauberian theorem (2.12)

$$D_{0+}^{\alpha-1} f(t) \sim f_1, \quad t \rightarrow 0+.$$

□

Remark 1 In Theorem 2, if $f_0 = 0$, then we can expand the order of fractional derivative α from $\frac{3}{2} < \alpha < 2$ to $1 < \alpha < 2$, and the oscillation process (2.13) has a unique solution f from $BSA_{2\alpha-1}(\mathbb{R}_+)$.

3 Multi-term Riemann-Liouville Fractional Oscillation Equations

In this section, we are concerned with the existence and uniqueness of solutions in $BSA_p(\mathbb{R}_+)$ to some classes of integro-differential equations with fractional order $1 < \alpha \leq 2$. We start with the following auxiliary lemma

Lemma 3 Let $a, b > 0$ and

$$f(x) = \frac{1}{ab^x + 1}, \quad g(x) = \left(\frac{1}{ab^{\frac{1}{x}} + 1} \right)^x, \quad h(x) = \left(\frac{1}{a(b^{\frac{1}{x}} + b^x) + 1} \right)^x.$$

Then

$$h(x) < f(x), \quad h(x) < g(x) \quad (3.1)$$

for every $x > 1$.

Proof We have

$$h(x) = \left(\frac{1}{a(b^{\frac{1}{x}} + b^x) + 1} \right)^x = \left(\frac{1}{(ab^{\frac{1}{x}} + 1) + ab^x} \right)^x < g(x)$$

and

$$\begin{aligned} h(x) &= \left(\frac{1}{a(b^{\frac{1}{x}} + b^x) + 1} \right)^x = \left(\frac{1}{(ab^x + 1) + ab^{\frac{1}{x}}} \right)^x \\ &< \left(\frac{1}{ab^x + 1} \right)^x < f(x). \end{aligned}$$

□

The following theorem plays a key role in the subsequent sections of this work.

Theorem 3 Let $k > 0$, $1 < \alpha_0 \leq 2$, and $g \in L^1(\mathbb{R}_+)$, such that $\|g\|_1 < \gamma(\alpha_0, k)$ with

$$\gamma(\alpha_0, k) = \begin{cases} \frac{|\tan(\alpha_0 \frac{\pi}{2})|}{1 + |\tan(\alpha_0 \frac{\pi}{2})|} k & , 1 < \alpha_0 < 2 \\ \frac{k}{\left\{ \frac{1+n\alpha_0}{k} \left[\left(\frac{k}{a_0 \sin(\alpha_1 \frac{\pi}{2})} \right)^{\frac{\alpha_n}{2}} + \left(\frac{k}{a_0 \sin(\alpha_1 \frac{\pi}{2})} \right)^{\frac{2}{\alpha_n}} \right] + 1 \right\}^{\frac{2}{\alpha_n}}} } & , \alpha_0 = 2, \end{cases} \quad (3.2)$$

$a_1, \dots, a_n > 0$, $a^0 = \max\{a_1, \dots, a_n\}$, $a_0 = \min\{a_1, \dots, a_n\}$, and $0 < \alpha_n < \dots < \alpha_1 < \alpha_0$, then the inverse Laplace transform

$$f := \mathcal{L}^{-1} \left(\frac{1}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)} \right) \quad (3.3)$$

is from $BSA_{2\alpha_0-1}(\mathbb{R}_+)$.

Proof Since $g \in L^1(\mathbb{R}_+)$, its Laplace transform $G(s)$ is holomorphic in the right half-plane, and therefore, $s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)$ is holomorphic in the right half-plane. We claim that the equation

$$s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s) = 0 \quad (3.4)$$

has no solution in the domain $\operatorname{Re} s > 0$. Assume the contrary that the Eq. (3.4) has a solution $s = r(\cos \varphi + i \sin \varphi)$, $|\varphi| < \frac{\pi}{2}$, then we would have

$$\begin{cases} r^{\alpha_0} \cos(\alpha_0 \varphi) + \sum_{j=1}^n a_j r^{\alpha_j} \cos(\alpha_j \varphi) + k + \operatorname{Re} G(s) = 0, \\ r^{\alpha_0} \sin(\alpha_0 \varphi) + \sum_{j=1}^n a_j r^{\alpha_j} \sin(\alpha_j \varphi) + \operatorname{Im} G(s) = 0. \end{cases} \quad (3.5)$$

From

$$k + \operatorname{Re} G(s) \geq k - |G(s)| \geq k - \|g\|_1 \geq k - \gamma(\alpha_0, k) > 0$$

and (3.5) we arrive at

$$r^{\alpha_0} \cos(\alpha_0 \varphi) + \sum_{j=1}^n a_j r^{\alpha_j} \cos(\alpha_j \varphi) = -(k + \operatorname{Re} G(s)) \leq \gamma(\alpha_0, k) - k < 0,$$

and thus one of the terms $\cos(\alpha_j \varphi)$, $j = 0, \dots, n$, must be negative. Since $\pi > \alpha_0 \varphi \geq \dots \geq \alpha_n \varphi \geq 0$ for $0 \leq \varphi < \frac{\pi}{2}$, then $\cos(\alpha_0 \varphi) \leq \dots \leq \cos(\alpha_n \varphi)$ for $0 \leq \varphi < \frac{\pi}{2}$. Consequently, there exists $0 \leq m \leq n$ such that

$$\cos\left(\frac{\pi}{2} \alpha_0\right) \leq \cos(\alpha_0 \varphi) \leq \dots \leq \cos(\alpha_m \varphi) \leq 0 \leq \cos(\alpha_{m+1} \varphi) \leq \dots \leq \cos(\alpha_n \varphi),$$

for $0 \leq \varphi < \frac{\pi}{2}$, and

$$\begin{aligned} & \left(r^{\alpha_0} + \sum_{j=1}^m a_j r^{\alpha_j} \right) \cos\left(\frac{\pi}{2} \alpha_0\right) \leq r^{\alpha_0} \cos(\alpha_0 \varphi) + \sum_{j=1}^m a_j r^{\alpha_j} \cos(\alpha_j \varphi) \\ & \leq - \sum_{j=m+1}^n a_j r^{\alpha_j} \cos(\alpha_j \varphi) + \gamma(\alpha_0, k) - k \leq \gamma(\alpha_0, k) - k. \end{aligned}$$

We arrive at

$$r^{\alpha_0} + \sum_{j=1}^m a_j r^{\alpha_j} > \frac{k - \gamma(\alpha_0, k)}{|\cos(\frac{\pi}{2} \alpha_0)|}. \quad (3.6)$$

For $0 \leq \varphi < \frac{\pi}{2}$ it is clear that $\sin(\alpha_m \varphi) > \dots > \sin(\alpha_1 \varphi) > \sin(\alpha_1 \frac{\pi}{2}) > \sin(\alpha_0 \frac{\pi}{2}) \geq 0$, and $\sin(\alpha_j \varphi) \geq 0$, $j = m+1, \dots, n$. From (3.5) we have

$$\begin{aligned} \gamma(\alpha_0, k) & > |G(s)| \geq -\operatorname{Im} G(s) \\ & = r^{\alpha_0} \sin(\alpha_0 \varphi) + \sum_{j=1}^m a_j r^{\alpha_j} \sin(\alpha_j \varphi) + \sum_{j=m+1}^n a_j r^{\alpha_j} \sin(\alpha_j \varphi) \\ & \geq r^{\alpha_0} \sin(\alpha_0 \varphi) + \sum_{j=1}^m a_j r^{\alpha_j} \sin(\alpha_j \varphi). \end{aligned} \quad (3.7)$$

We distinguish the following two cases:

Case $\alpha_0 < 2$: From (3.5) and (3.7) we have

$$\gamma(\alpha_0, k) > (r^{\alpha_0} + \sum_{j=1}^m a_j r^{\alpha_j}) \sin\left(\alpha_0 \frac{\pi}{2}\right) > \frac{(k - \gamma(\alpha_0, k)) |\sin(\alpha_0 \frac{\pi}{2})|}{|\cos(\frac{\pi}{2} \alpha_0)|}.$$

As such, we get

$$\gamma(\alpha_0, k) > (k - \gamma(\alpha_0, k)) \left| \tan\left(\alpha_0 \frac{\pi}{2}\right) \right|,$$

or,

$$\gamma(\alpha_0, k) > \frac{|\tan(\alpha_0 \frac{\pi}{2})|}{1 + |\tan(\alpha_0 \frac{\pi}{2})|} k,$$

that contradicts the definition of $\gamma(\alpha_0, k)$. Thus, (3.4) has no solution in the first quarter. By the complex conjugate theorem, (3.4) has no solution in the fourth quarter.

Case $\alpha_0 = 2$: From (3.5) and (3.7) we have

$$\begin{cases} r^2 + \sum_{j=1}^m a_j r^{\alpha_j} > k - \gamma(2, k) \\ \gamma(2, k) > \sin\left(\alpha_1 \frac{\pi}{2}\right) \sum_{j=1}^m a_j r^{\alpha_j}. \end{cases} \quad (3.8)$$

Now we consider two subcases:

1. $r \geq 1$. From (3.8) and $r^2 \geq r^{\alpha_1} \geq \dots \geq r^{\alpha_n}$ we have

$$\begin{cases} r^2(1 + na^0) > k - \gamma(2, k) \\ \gamma(2, k) > a_0 \sin\left(\alpha_1 \frac{\pi}{2}\right) r^{\alpha_n} \end{cases}$$

then, because $\gamma(2, k) < k$,

$$\begin{aligned} \gamma(2, k) k^{\frac{2}{\alpha_n}-1} &> (\gamma(2, k))^{\frac{2}{\alpha_n}} > \left(a_0 \sin\left(\alpha_1 \frac{\pi}{2}\right)\right)^{\frac{2}{\alpha_n}} r^2 \\ &> \left(a_0 \sin\left(\alpha_1 \frac{\pi}{2}\right)\right)^{\frac{2}{\alpha_n}} \frac{k - \gamma(2, k)}{(1 + na^0)}. \end{aligned}$$

As such, we get

$$\begin{aligned} \gamma(2, k) &> \frac{\left(a_0 \sin\left(\alpha_1 \frac{\pi}{2}\right)\right)^{\frac{2}{\alpha_n}} k}{(1 + na^0)k^{\frac{2}{\alpha_n}-1} + \left(a_0 \sin\left(\alpha_1 \frac{\pi}{2}\right)\right)^{\frac{2}{\alpha_n}}} \\ &= \frac{k}{\frac{1+na^0}{k} \left(\frac{k}{a_0 \sin\left(\alpha_1 \frac{\pi}{2}\right)}\right)^{\frac{2}{\alpha_n}} + 1}. \end{aligned}$$

By Lemma 3 we get

$$\gamma(2, k) > \frac{k}{\left\{ \frac{1+na^0}{k} \left[\left(\frac{k}{a_0 \sin\left(\alpha_1 \frac{\pi}{2}\right)}\right)^{\frac{\alpha_n}{2}} + \left(\frac{k}{a_0 \sin\left(\alpha_1 \frac{\pi}{2}\right)}\right)^{\frac{2}{\alpha_n}} \right] + 1 \right\}^{\frac{2}{\alpha_n}}}$$

that contradicts the definition of $\gamma(2, k)$.

2. $0 < r < 1$. From (3.8) and $r^2 < r^{\alpha_1} < \dots < r^{\alpha_n}$, we get

$$\begin{cases} r^{\alpha_n}(1 + na^0) > k - \gamma(2, k) \\ \gamma(2, k) > a_0 \sin\left(\alpha_1 \frac{\pi}{2}\right) r^2, \end{cases}$$

and because $\gamma(2, k) < (\gamma(2, k))^{\alpha_n/2} k^{1-\alpha_n/2}$, then

$$\begin{aligned} (\gamma(2, k))^{\frac{\alpha_n}{2}} &> \left(a_0 \sin\left(\alpha_1 \frac{\pi}{2}\right)\right)^{\frac{\alpha_n}{2}} r^{\alpha_n} > \left(a_0 \sin\left(\alpha_1 \frac{\pi}{2}\right)\right)^{\frac{\alpha_n}{2}} \frac{k - \gamma(2, k)}{(1 + na^0)} \\ &> \left(a_0 \sin\left(\alpha_1 \frac{\pi}{2}\right)\right)^{\frac{\alpha_n}{2}} \frac{k - (\gamma(2, k))^{\frac{\alpha_n}{2}} k^{1-\alpha_n/2}}{(1 + na^0)}. \end{aligned}$$

As such, we get

$$(\gamma(2, k))^{\frac{\alpha_n}{2}} > \frac{(a_0 \sin(\alpha_1 \frac{\pi}{2}))^{\frac{\alpha_n}{2}} k^{\frac{\alpha_n}{2}}}{(1 + na^0)k^{\frac{\alpha_n}{2}-1} + (a_0 \sin(\alpha_1 \frac{\pi}{2}))^{\frac{\alpha_n}{2}}},$$

that is

$$\begin{aligned} \gamma(2, k) &> \left(\frac{(a_0 \sin(\alpha_1 \frac{\pi}{2}))^{\frac{\alpha_n}{2}} k^{\frac{\alpha_n}{2}}}{(1 + na^0)k^{\frac{\alpha_n}{2}-1} + (a_0 \sin(\alpha_1 \frac{\pi}{2}))^{\frac{\alpha_n}{2}}} \right)^{\frac{2}{\alpha_n}} \\ &= \frac{k}{\left[\frac{1+na^0}{k} \left(\frac{k}{a_0 \sin(\alpha_1 \frac{\pi}{2})} \right)^{\frac{\alpha_n}{2}} + 1 \right]^{\frac{2}{\alpha_n}}}. \end{aligned}$$

By Lemma 3 we get

$$\gamma(2, k) > \frac{k}{\left\{ \frac{1+na^0}{k} \left[\left(\frac{k}{a_0 \sin(\alpha_1 \frac{\pi}{2})} \right)^{\frac{\alpha_n}{2}} + \left(\frac{k}{a_0 \sin(\alpha_1 \frac{\pi}{2})} \right)^{\frac{2}{\alpha_n}} \right] + 1 \right\}^{\frac{2}{\alpha_n}}}$$

that contradicts the definition of $\gamma(2, k)$.

Thus, (3.4) has no solution in the first quarter. By the complex conjugate theorem, (3.4) has no solution in the fourth quarter.

Consequently, (3.4) has no solution in the right half plane, and

$$\frac{1}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)}$$

is holomorphic in the domain $\operatorname{Re} s > 0$.

Next, we will show that there exists $C > 0$ such that

$$\left| s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s) \right| > C |s|^{\alpha_0}, \quad \operatorname{Re} s > 0, \quad 1 < \alpha_0 \leq 2. \quad (3.9)$$

Indeed, we consider the following function

$$\phi(s) := \frac{\left| s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s) \right|}{|s|^{\alpha_0}}, \quad \operatorname{Re} s > 0.$$

Due to $|G(s)| < \gamma(\alpha_0, k)$, we have

$$\lim_{|s| \rightarrow \infty} \phi(s) = 1, \quad \operatorname{Re} s > 0.$$

Therefore, there exist $N, C_1 > 0$, such that if $|s| > N$, $\operatorname{Re} s > 0$, then $\phi(s) > C_1$. On the other hand, we have

$$\lim_{|s| \rightarrow 0} \phi(s) = \infty, \quad \operatorname{Re} s > 0.$$

Consequently, there exist $\delta, C_2 > 0$, such that if $|s| < \delta$, $\operatorname{Re} s > 0$, then $\phi(s) > C_2$. Since $\phi(s)$ is continuous in the domain $\{s \mid \operatorname{Re} s \geq 0, s \neq 0\}$, it is bounded over the compact set $B = \{s \mid |s| \in [\delta, N], \operatorname{Re} s \geq 0\}$, so we obtain

$$\min_{s \in B} \phi(s) = C_3 > 0.$$

Denote $C = \min\{C_1, C_2, C_3\}$, then $C > 0$ and

$$\phi(s) > C > 0, \quad \operatorname{Re} s > 0.$$

Consequently, (3.9) holds, and by Corollary 1 we obtain $f \in BSA_{2\alpha_0-1}(\mathbb{R}_+)$, and Theorem 3 is proved. $\square$

Combining Corollary 1 and Theorem 3 we arrive at

Corollary 3 *Let conditions of Theorem 3 hold, and $\beta < \alpha_0 - \frac{1}{2}$, then*

$$\mathcal{L}^{-1} \left(\frac{s^\beta}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)} \right) \in BSA_{2(\alpha_0-\beta)-1}(\mathbb{R}_+). \quad (3.10)$$

As an example consider the Prabhakar (three-parametric Mittag-Leffler) function [9]

$$E_{\alpha, \beta}^\gamma(z) = \sum_{j=0}^{\infty} \frac{(\gamma)_j}{j! \Gamma(\alpha j + \beta)} z^j, \quad \operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0, \gamma > 0,$$

where $(\gamma)_j = \gamma(\gamma+1) \cdots (\gamma+j-1)$.

The Laplace transform of the Prabhakar function has the following form [9]

$$\mathcal{L}(t^{\beta-1} E_{\alpha, \beta}^\gamma(\lambda t^\alpha))(s) = \frac{s^{\alpha\gamma-\beta}}{(s^\alpha - \lambda)^\gamma},$$

which is valid for all $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \beta > 0$, $\gamma > 0$, $\lambda \in \mathbb{C}$, such that $|\lambda s^{-\alpha}| < 1$.

Consequently, by Corollary 3, if k , $\alpha > 0$, and $\beta > \frac{1}{2}$, then $t^{\beta-1} E_{\alpha,\beta}^{\gamma}(-kt^{\alpha}) \in BSA_{2\beta-1}(\mathbb{R}_+)$.

Consider now the multi-term Riemann-Liouville fractional integro-differential equation

$$\begin{aligned} D_{0+}^{\alpha_0} f(t) + \sum_{j=1}^n a_j D_{0+}^{\alpha_j} f(t) + k f(t) + \int_0^t g(t-\tau) f(\tau) d\tau &= h(t) \\ I_{0+}^{2-\alpha_0} f(0+) &= f_0, \quad D_{0+}^{\alpha_0-1} f(0+) = f_1, \end{aligned} \quad (3.11)$$

$$0 < \alpha_n < \cdots < \alpha_1 \leq \alpha_0 - 1 \leq 1, \quad \alpha_0 > \frac{3}{2},$$

where $k, a_1, a_2, \dots, a_n \in \mathbb{R}_+$, $g, h \in L^1(\mathbb{R}_+)$ are given, and f is the unknown. Special cases of (3.11) have been considered in [10].

Now, from Corollary 3, we derive conditions to ensure the existence and uniqueness of solutions in $BSA_p(\mathbb{R}_+)$ for the problem (3.11).

Theorem 4 Let $f_0, f_1 \in \mathbb{R}$, $g, h \in L^1(\mathbb{R}_+)$, be given, and $\|g\|_1 < \gamma(\alpha_0, k)$ with $\gamma(\alpha_0, k)$ being defined by (3.2). Then the multi-term Riemann-Liouville fractional integro-differential equation (3.11) has a unique solution f from $BSA_{2\alpha_0-1}(\mathbb{R}_+)$.

Proof Since $I_{0+}^{2-\alpha_0} f(0+) = f_0$, and $2-\alpha_0 \leq 1-\alpha_1$, $2-\alpha_0 < 1-\alpha_j$, $j = 2, \dots, n$, then $I_{0+}^{1-\alpha_j} f(0+) = 0$, $j = 2, \dots, n$, and $I_{0+}^{1-\alpha_1} f(0+) = 0$, if $\alpha_1 < \alpha_0 - 1$, and $I_{0+}^{1-\alpha_1} f(0+) = f_0$, if $\alpha_1 = \alpha_0 - 1$.

Consequently, applying the Laplace transform to Eq. (3.11) and taking into account [10]

$$\mathcal{L}(D_{0+}^{\alpha} f)(s) = s^{\alpha} F(s) - \sum_{k=0}^{m-1} s^{m-k-1} D_{0+}^{\alpha+k-m} f(0+), \quad m-1 < \alpha \leq m, \quad (3.12)$$

we obtain

$$\begin{aligned} H(s) &= \left(s^{\alpha_0} F(s) - s I_{0+}^{2-\alpha_0} f(0+) - D_{0+}^{\alpha_0-1} f(0+) \right) \\ &\quad + \sum_{j=1}^n a_j \left(s^{\alpha_j} F(s) - I_{0+}^{1-\alpha_j} f(0+) \right) + k F(s) + G(s) F(s) \\ &= s^{\alpha_0} F(s) - s f_0 - f_1 + \sum_{j=1}^n a_j s^{\alpha_j} F(s) - a_1 f_0 \mu + k F(s) + G(s) F(s) \end{aligned} \quad (3.13)$$

where

$$\mu = \begin{cases} 1, & \text{if } \alpha_0 - \alpha_1 = 1 \\ 0, & \text{if } \alpha_0 - \alpha_1 > 1 \end{cases}.$$

Solving for $F(s)$ yields

$$F(s) = \frac{sf_0 + f_1 + a_1 f_0 \mu + H(s)}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)}. \quad (3.14)$$

Denote

$$\begin{aligned} W(s) &= \frac{1}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)}, \\ W_1(s) &= \frac{s}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)} \end{aligned} \quad (3.15)$$

then according to Corollary 3, their inverse Laplace transforms, namely $w(t)$ and $w_1(t)$, belong to $BSA_{2\alpha_0-1}(\mathbb{R}_+)$ and $BSA_{2\alpha_0-3}(\mathbb{R}_+) \subset BSA_{2\alpha_0-1}(\mathbb{R}_+)$, and

$$f(t) = (f_1 + a_1 f_0 \mu) w(t) + f_0 w_1(t) + \int_0^t w(t - \tau) h(\tau) d\tau. \quad (3.16)$$

Since $w \in BSA_{2\alpha_0-1}(\mathbb{R}_+)$ and $h \in L^1(\mathbb{R}_+)$, by Lemma 1, their Laplace convolution $w * h$ belongs to $BSA_{2\alpha_0-1}(\mathbb{R}_+)$. Hence, f , defined by (3.16), is from $BSA_{2\alpha_0-1}(\mathbb{R}_+)$. As $G(s), H(s) \rightarrow 0$ as $\operatorname{Re} s \rightarrow \infty$, using the Tauberian theorem for the Laplace transform (2.12) we have

$$F(s) \sim \frac{f_0}{s^{\alpha_0-1}}, \quad \operatorname{Re} s \rightarrow \infty \implies f(t) \sim \frac{t^{\alpha_0-2}}{\Gamma(\alpha_0-1)}, \quad t \rightarrow 0+. \quad (3.17)$$

Consequently, $I_{0+}^{2-\alpha_0} f(0+) = f_0$. Next,

$$\begin{aligned} (\mathcal{L} D_{0+}^{\alpha_0-1} f)(s) &= s^{\alpha_0-1} F(s) - I_{0+}^{2-\alpha_0} f(0+) \\ &= \frac{s^{\alpha_0} f_0 + s^{\alpha_0-1} (f_1 + a_1 f_0 \mu) + s^{\alpha_0-1} H(s)}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)} - f_0 \\ &= \frac{s^{\alpha_0-1} (f_1 + a_1 f_0 \mu) + s^{\alpha_0-1} H(s) - f_0 \left(\sum_{j=1}^n a_j s^{\alpha_j} + k + G(s) \right)}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)} \sim \frac{f_1}{s} \end{aligned}$$

as $\operatorname{Re} s \rightarrow \infty$. Consequently, again by the Tauberian theorem (2.12)

$$D_{0+}^{\alpha_0-1} f(t) \sim f_1, \quad t \rightarrow 0+.$$

Conversely, let f be given by (3.16), where w and w_1 are defined as the Laplace inverses of (3.15). Then $f \in BSA_{2\alpha_0-1}(\mathbb{R}_+)$, and $I_{0+}^{2-\alpha_0} f(0+) = f_0$, $D_{0+}^{\alpha_0-1} f(0+) = f_1$. Applying the Laplace transform to (3.16) and taking into account (3.15) we arrive at (3.14). Hence, (3.13) holds. The Laplace inverse of (3.13) yields (3.11). $\square$

Remark 2 In Theorem 4, if $f_0 = 0$, then we can extend the range of α_0 from $(\frac{3}{2}, 2]$ to $(1, 2]$, and the Eq. (3.11) has a unique solution f from $BSA_{2\alpha_0-1}(\mathbb{R}_+)$.

4 Multi-Term Caputo Fractional Oscillation Equation

Consider the Caputo fractional integro-differential equation

$$\begin{aligned} {}^C\partial_t^{\alpha_0} f(t) + \sum_{j=1}^n a_j {}^C\partial_t^{\alpha_j} f(t) + kf(t) + \int_0^t g(t-\tau)f(\tau)d\tau &= h(t), \\ f(0+) = f_0, f'(0+) &= f_1, \end{aligned} \quad (4.1)$$

$$0 < \alpha_n < \dots < \alpha_{l+1} < 1 < \alpha_l < \dots < \alpha_1 < \alpha_0 \leq 2,$$

where $a_j, k \in \mathbb{R}_+$, $j = 1, 2, \dots, n$, and $g, h \in L^1(\mathbb{R}_+)$ are given, and f is the unknown.

In the following theorem, we derive conditions to ensure the existence and uniqueness of solutions in $BSA_p(\mathbb{R}_+)$ for the initial-value problem (4.1). Our result is based on Corollary 3.

Theorem 5 Let $f_0, f_1 \in \mathbb{R}$, $g, h \in L^1(\mathbb{R}_+)$, be given, and $\|g\|_1 < \gamma(\alpha_0, k)$ with $\gamma(\alpha_0, k)$ being defined by (3.2). Then the Caputo fractional integro-differential equation (4.1) has a unique solution f from $BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$.

Proof Applying the Laplace transform to Eq. (4.1) and taking into account [10] that

$$\mathcal{L}\left({}^C\partial_t^{\alpha} f\right)(s) = s^{\alpha} F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0+), \quad m-1 < \alpha \leq m. \quad (4.2)$$

(4.2) we obtain

$$\begin{aligned} (s^{\alpha_0} F(s) - s^{\alpha_0-1} f_0 - s^{\alpha_0-2} f_1) + \sum_{j=1}^l a_j (s^{\alpha_j} F(s) - s^{\alpha_j-1} f_0 - s^{\alpha_j-2} f_1) \\ + \sum_{j=l+1}^n a_j (s^{\alpha_j} F(s) - s^{\alpha_j-1} f_0) + kF(s) + G(s)F(s) = H(s). \end{aligned} \quad (4.3)$$

Solving for $F(s)$ yields

$$F(s) = \frac{s^{\alpha_0-1} f_0 + s^{\alpha_0-2} f_1 + f_0 \sum_{j=1}^n a_j s^{\alpha_j-1} + f_1 \sum_{j=1}^l a_j s^{\alpha_j-2} + H(s)}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)}. \quad (4.4)$$

Denote

$$\begin{aligned} Q_j(s) &= \frac{s^{\alpha_j-1}}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)}, \quad j = 0, 1, \dots, n, \\ R_j(s) &= \frac{s^{\alpha_j-2}}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)}, \quad j = 0, 1, \dots, l, \end{aligned} \quad (4.5)$$

and $W(s)$ by (3.15), then according to Corollary 3, their inverse Laplace transforms, namely q_j , r_j , and w , belong to $BSA_{2(\alpha_0-\alpha_j)+1}(\mathbb{R}_+)$
 $\subset BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$, $BSA_{2(\alpha_0-\alpha_j)+3}(\mathbb{R}_+) \subset BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$, and
 $BSA_{2\alpha_0-1}(\mathbb{R}_+)$, respectively. Moreover,

$$f(t) = f_0 q_0(t) + f_1 r_0(t) + f_0 \sum_{j=1}^n a_j q_j(t) + f_1 \sum_{j=1}^l a_j r_j(t) + \int_0^t w(t-\tau) h(\tau) d\tau. \quad (4.6)$$

Since $w \in BSA_{2\alpha_0-1}(\mathbb{R}_+)$ and $h \in L^1(\mathbb{R}_+)$, by Lemma 1, their Laplace convolution $w * h$ belongs to $BSA_{2\alpha_0-1}(\mathbb{R}_+) \subset BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$. Hence, f , defined by (4.6), is from $BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$. Using the Tauberian theorem for the Laplace transform (2.12) we have

$$F(s) \sim \frac{f_0}{s}, \quad \operatorname{Re} s \rightarrow \infty \implies f(t) \sim f_0, \quad t \rightarrow 0+.$$

Consequently, $f(0+) = f_0$.

Next, as $\operatorname{Re} s \rightarrow \infty$,

$$\begin{aligned} (\mathcal{L}f')(s) &= sF(s) - f(0) \\ &= \frac{s^{\alpha_0} f_0 + s^{\alpha_0-1} f_1 + f_0 \sum_{j=1}^n a_j s^{\alpha_j} + f_1 \sum_{j=1}^l a_j s^{\alpha_j-1} + sH(s)}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)} - f_0 \\ &= \frac{s^{\alpha_0-1} f_1 + f_1 \sum_{j=1}^l a_j s^{\alpha_j-1} + sH(s) - f_0 k - f_0 G(s)}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + k + G(s)} \\ &\sim \frac{f_1}{s}, \quad \operatorname{Re} s \rightarrow \infty \implies f'(t) \sim f_1, \quad t \rightarrow 0+. \end{aligned}$$

Conversely, let f be given by (4.6), where q_j , r_j , and w are defined as the Laplace inverse transforms of (4.5) and (3.15). Then $f \in BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$ and $f(0+) =$

$f_0, f'(0+) = f_1$. Applying the Laplace transform to (4.6) and taking into account (4.5) and (3.15) we arrive at (4.4). Hence, (4.3) holds. The Laplace inverse transform of (4.3) yields (4.1). $\square$

Remark 3 Although $g, h \in L^1(\mathbb{R}_+)$, in general $f \notin L^1(\mathbb{R}_+)$. In fact, if $f \in L^1(\mathbb{R}_+)$, then $|F(s)| \leq \|f\|_1$ for $\operatorname{Re} s \geq 0$. But if $f_0 \neq 0$, then from 4.4 we would have

$$F(s) \sim Cs^{\alpha_n-2} \rightarrow \infty,$$

as $s \rightarrow 0+$. Consequently, $f \notin L^1(\mathbb{R}_+)$.

5 Mixed Caputo Riemann-Liouville Fractional Oscillation Equation

Now we consider the following mixed Caputo Riemann-Liouville fractional integro-differential equation with a dominant Caputo fractional derivative

$$\begin{aligned} & {}^C\partial_t^{\alpha_0} f(t) + \sum_{j=1}^n a_j {}^C\partial_t^{\alpha_j} f(t) + \sum_{j=1}^m b_j D_{0+}^{\beta_j} f(t) + kf(t) \\ & + \int_0^t g(t-\tau)f(\tau)d\tau = h(t), \\ & f(0+) = f_0, \quad f'(0+) = f_1, \end{aligned} \tag{5.1}$$

$$\begin{aligned} & 0 < \alpha_n < \cdots < \alpha_{l+1} < 1 < \alpha_l < \cdots < \alpha_1 < \alpha_0 \leq 2, \\ & 0 < \beta_m < \cdots < \beta_1 < 1, \end{aligned}$$

where $g, h \in L^1(\mathbb{R}_+)$, $a_1, \dots, a_n, b_1, \dots, b_m, k \in \mathbb{R}_+$, are given, and f is the unknown.

In the literature, there are many papers investigating existence and uniqueness results for fractional differential equations which contain mixed Riemann-Liouville and Caputo fractional derivatives, see [14, 15, 25]. But we do not find any work dealing with the fractional differential equation in the form (5.1). Motivated by this fact, and to fill this gap, in this work we investigate the existence and uniqueness of solutions in $BSA_p(\mathbb{R}_+)$ for the problem (5.1).

The following theorem state the existence and uniqueness result concerning solutions in $BSA_p(\mathbb{R}_+)$ of the problem (5.1).

Theorem 6 Let $f_0, f_1 \in \mathbb{R}$, $g, h \in L^1(\mathbb{R}_+)$, be given, and $\|g\|_1 < \gamma(\alpha_0, k)$ with $\gamma(\alpha_0, k)$ being defined by (3.2). Then the mixed Caputo Riemann-Liouville fractional integro-differential equation (5.1) has a unique solution f from $BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$.

Proof Since $f(0+) = f_0$ then $I_{0+}^{1-\beta_j} f(0+) = 0, j = 1, \dots, m$. Applying the Laplace transform to Eq. (5.1) and taking into account (4.2) and (3.12) we obtain

$$\begin{aligned}
 H(s) &= (s^{\alpha_0} F(s) - s^{\alpha_0-1} f(0+) - s^{\alpha_0-2} f'(0+)) \\
 &\quad + \sum_{j=1}^l a_j (s^{\alpha_j} F(s) - s^{\alpha_j-1} f(0+) - s^{\alpha_j-2} f'(0+)) \\
 &\quad + \sum_{j=l+1}^n a_j (s^{\alpha_j} F(s) - s^{\alpha_j-1} f(0+)) + \sum_{j=1}^m b_j \left(s^{\beta_j} F(s) - I_{0+}^{1-\beta_j} f(0+) \right) \\
 &\quad + kF(s) + G(s)F(s) \\
 &= (s^{\alpha_0} F(s) - s^{\alpha_0-1} f_0 - s^{\alpha_0-2} f_1) + \sum_{j=1}^l a_j (s^{\alpha_j} F(s) - s^{\alpha_j-1} f_0 - s^{\alpha_j-2} f_1) \\
 &\quad + \sum_{j=l+1}^n a_j (s^{\alpha_j} F(s) - s^{\alpha_j-1} f_0) \\
 &\quad + \sum_{j=1}^m b_j s^{\beta_j} F(s) + kF(s) + G(s)F(s). \tag{5.2}
 \end{aligned}$$

Solving for $F(s)$ yields

$$F(s) = \frac{f_0 s^{\alpha_0-1} + f_1 s^{\alpha_0-2} + f_0 \sum_{j=1}^n a_j s^{\alpha_j-1} + f_1 \sum_{j=1}^l a_j s^{\alpha_j-2} + H(s)}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + \sum_{j=1}^m b_j s^{\beta_j} + k + G(s)}. \tag{5.3}$$

Denote

$$\begin{aligned}
 X_j(s) &= \frac{s^{\alpha_j-1}}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + \sum_{j=1}^m b_j s^{\beta_j} + k + G(s)}, \quad j = 0, 1, \dots, n, \\
 Y_j(s) &= \frac{s^{\alpha_j-2}}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + \sum_{j=1}^m b_j s^{\beta_j} + k + G(s)}, \quad j = 0, 1, \dots, l, \\
 Z(s) &= \frac{1}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + \sum_{j=1}^m b_j s^{\beta_j} + k + G(s)}, \tag{5.4}
 \end{aligned}$$

then according to Corollary 3, their inverse Laplace transforms, namely x_j , y_j , and z , belong to $BSA_{2(\alpha_0-\alpha_j)+1}(\mathbb{R}_+) \subset BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$, $BSA_{2(\alpha_0-\alpha_j)+3}(\mathbb{R}_+) \subset BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$, and $BSA_{2\alpha_0-1}(\mathbb{R}_+)$, respectively. Moreover,

$$f(t) = f_0 x_0(t) + f_1 y_0(t) + f_0 \sum_{j=1}^n a_j x_j(t) + f_1 \sum_{j=1}^l a_j y_j(t) + \int_0^t z(t-\tau) h(\tau) d\tau. \tag{5.5}$$

Since $z \in BSA_{2\alpha_0-1}(\mathbb{R}_+)$, and $h \in L^1(\mathbb{R}_+)$, by Lemma 1, their Laplace convolution $z * h$ belongs to $BSA_{2\alpha_0-1}(\mathbb{R}_+) \subset BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$. Hence, f , defined by (5.5), is from $BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$. From (5.3) we have

$$F(s) \sim \frac{f_0}{s}, \quad \operatorname{Re} s \rightarrow \infty.$$

Using the Tauberian theorem for the Laplace transform (2.12) we obtain

$$f(t) \sim f_0, \quad t \rightarrow 0+.$$

Consequently, $f(0+) = f_0$. Next, as $\operatorname{Re} s \rightarrow \infty$,

$$\begin{aligned} (\mathcal{L}f')(s) &= sF(s) - f(0+) \\ &= \frac{s^{\alpha_0}f_0 + s^{\alpha_0-1}f_1 + f_0 \sum_{j=1}^n a_j s^{\alpha_j} + f_1 \sum_{j=1}^m a_j s^{\alpha_j-1} + sH(s)}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + \sum_{j=1}^m b_j s^{\beta_j} + k + G(s)} - f_0 \\ &= \frac{s^{\alpha_0-1}f_1 + f_1 \sum_{j=1}^m a_j s^{\alpha_j-1} - f_0 \sum_{j=1}^m b_j s^{\beta_j} + sH(s) - f_0 k - f_0 G(s)}{s^{\alpha_0} + \sum_{j=1}^n a_j s^{\alpha_j} + \sum_{j=1}^m b_j s^{\beta_j} + k + G(s)} \\ &\sim \frac{f_1}{s}, \quad \operatorname{Re} s \rightarrow \infty \implies f'(t) \sim f_1, \quad t \rightarrow 0+. \end{aligned}$$

Conversely, let f be given by (5.5), where x_j , y_j , and z are defined as the Laplace inverse transforms of (5.5). Then $f \in BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$ and $f(0+) = f_0$, $f'(0+) = f_1$. Applying the Laplace transform to (5.5) and taking into account (5.4) we arrive at (5.3). Hence, (5.2) holds. The Laplace inverse transform of (5.2) yields (5.1). $\square$

6 Conclusions

In this work, we derive conditions to ensure the existence and uniqueness of solutions in $BSA_p(\mathbb{R}_+)$ for integro-differential equations of order $1 < \alpha \leq 2$. Surprisingly, multi-term Caputo fractional oscillation equations, multi-term Riemann-Liouville fractional equations, and mixed Caputo Riemann-Liouville fractional equations all have solutions in $BSA_p(\mathbb{R}_+)$ spaces.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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